## The process of soiling and the life of bank notes in the Netherlands

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# The Process of Soiling and the Life of Bank Notes in the Netherlands

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Summary

This paper presents a statistical method for estimating probabilities of soiling and withdrawal of bank notes. The method is applied to seven circulation trials of Dutch bank notes of different denominations. The best fitting distribution for the soiling process is selected for each trial and its parameters are estimated. Then the average soiling time and the average life of the notes are calculated. A notable aspect of our analysis is that large differences with respect to the soiling process were found among notes of one and the same denomination in different circulation trials.

*Keywords*: LIFE OF BANKNOTES; LIFE TABLE ANALYSIS; CIRCULATION TRIALS; NON-LINEAR ESTIMATION; GRADUATION OF DISTRIBUTIONS

## 1. Introduction

After its conception in the paper mill and the printing works, the life of a Dutch bank note starts at the moment it is brought into circulation by the Netherlands Bank. Then begins a wandering through the hands of market vendors, trouserpockets, purses, handbags, paying-counters and private vaults. During this circulation the bank note becomes soiled. One day it will come to the counter of a commercial bank, a savings bank or a post-office from where it is not issued again, but returned to the central bank.

A bank note returning to the Netherlands Bank is sorted mechanically to see if it is fit or unfit for further circulation. A soiled bank note is destroyed and replaced by a new one. Hence, its life has ended. An approved note, however, is again brought into circulation and its life continues until its final return to the Bank.

In 1980 the Netherlands Bank sorted 663 million notes, of which 232 million were rejected and replaced. This amounted to 93 per cent of the total issue of new bank notes in that year.

It is evident that an insight into the processes of soiling and withdrawal is imperative to the Bank's operations. The average life of bank notes is one of the determinants of the future supply of bank notes and thus of the printing capacity and the demand for bank note paper (see Fase, Van der Hoeven and Van Nieuwkerk, 1979).

Usually the average life of bank notes is measured in a model which describes the rejection of notes directly by some probability distribution (see Coenen, 1971; Gillieson, 1977; Koeze, 1979, 1980). However, this simple life model of bank notes has some serious drawbacks. Firstly, soiled bank notes do not return to the Bank directly but only after they have come to the counters of commercial banks, etc. Therefore the simple model simultaneously describes two processes, namely the process of soiling and the process of withdrawal. Secondly, the samples at our disposal for quantifying the simple life model of bank notes are so large (from 100 000 to 600 000 notes) that every test of the hypothesis that bank notes deteriorate according to a specific probability distribution invariably yields a negative result. This is mainly the results of fluctuations in the withdrawal process.



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This paper develops a statistical method which distinguishes between the processes of soiling and withdrawal, and assumes that the soiling process by itself is based on some (simple) probability distribution, by contrast with the soiling process plus the process of withdrawal. In this way we are able to calculate the average soiling time, i.e. the life which bank notes would have had if they were withdrawn from circulation as soon as they had become soiled. In addition we can calculate how long it takes, on average, before soiled notes are withdrawn by the Bank. Together, these two figures add up to the actual life of bank notes.

## 2. The Method

When bank notes are sorted mechanically to see if they are soiled or unsoiled, their numbers are registered and recorded on magnetic tape. This sorting system, which is unique in the world, enables us to monitor the receipts and rejects of specific series, which have been designated beforehand, in the course of time. Hence, such a circulation trial with, say, $N$ bank notes yield information for each period on the number of bank notes *returned* ($o_t$) and the number of notes *rejected* ($v_t$). Here $t = 1, \ldots, T^*$ with all notes of the trial series brought into circulation at the same moment ($t = 0$) and with $T^*$ the length of the trial period.

In the usual life model the *average life* ($l^+$) calculated directly with the help of the probability of failure $p_t^+$, i.e. the probability that a bank note is rejected in period $t$:

$$l^+ = \sum_{t=1}^{T'} (t-1)\, p_t^+ = \sum_{t=1}^{T'} S_t^+ \quad \text{with} \quad S_t^+ = 1 - \sum_{\tau=1}^{t} p_\tau^+. \tag{1}$$

Here $T'$ is the moment that all bank notes have been withdrawn and $S_t^+$ the *probability of survival*, i.e. the probability that a note has not been rejected at the end of period $t$. When calculating the average life along these lines, $v_t/N$ is used as estimate for $p_t^+$. Since our samples are very large, throughout this paper we will ignore sample variation and set $p_t^+ = v_t/N$.

This calculation above only uses information on the number of notes rejected ($v_t$). However, the registration of the number of notes received ($o_t$) enables us to distinguish between the process of soiling and that of withdrawal. Analogous to the failure rate we define the *probability of soiling* $p_t$ as the probability that a bank note in the $t$th period of circulation becomes soiled according to the criterion set by the Bank. Then the *average soiling time* ($l$) can be calculated as

$$l = \sum_{t=1}^{T} (t-1)\, p_t = \sum_{t=1}^{T} S_t \quad \text{with} \quad S_t = 1 - \sum_{\tau=1}^{t} p_\tau. \tag{2}$$

Here $T$ is the moment that all bank notes are soiled ($T \leqslant T'$) and $S_t$ is the analogue of the probability of survival which we call the *probability of remaining unsoiled*.

The difference between average life and soiling time ($l^+ - l$) indicates how long it takes, on average, for soiled bank notes to be returned to the Bank and withdrawn from circulation. Thus, given $p_t$ the process of soiling can be separated from the withdrawal process.

In order to estimate the probability of soiling $p_t = n_t/N$, where $n_t$ is the (unknown) number of bank notes which become soiled in period $t$ and where, again, we ignore sample variation, we assume that the proportion of soiled notes in circulation is the same as that for notes returned to the Bank, i.e. $v_t/o_t$. Then Table 1 shows how we derive an estimate for $p_t$ from $o_t$ and $v_t$.

Our assumption is that [3]/[2] = [4]/[5] so that

$$p_1 = v_1/o_1,$$
$$p_2 = (v_2/o_2)(N_1/N) - v_1/o_1 + v_1/N$$

and in general

$$p_t = (v_t/o_t)(N_{t-1}/N) - (v_{t-1}/o_{t-1})(N_{t-2}/N) + v_{t-1}/N. \tag{3}$$

TABLE 1
*Circulation and soiling of a trial of bank notes*

| Period | [1] Number of notes that become soiled | [2] Number of notes received | [3] Number of notes rejected | [4] Number of soiled notes in circulation | [5] Total number of notes in circulation (before rejection) |
|---|---|---|---|---|---|
| 1 | $n_1$ | $o_1$ | $v_1$ | $n_1$ | $N$ |
| 2 | $n_2$ | $o_2$ | $v_2$ | $n_1-v_1+n_2$ | $N-v_1=N_1$ |
| 3 | $n_3$ | $o_3$ | $v_3$ | $(n_1-v_1)+(n_2-v_2)+n_3$ | $N_1-v_2=N_2$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $t$ | $n_t$ | $o_t$ | $v_t$ | $\sum_{\tau=1}^{t-1}(n_\tau-v_\tau)+n_t$ | $N_{t-2}-v_{t-1}=N_{t-1}$ |

Koeze (1979, 1980) shows that our assumption [3]/[2] = [4]/[5] is a realistic one because soiled bank notes do not return to the Bank faster than unsoiled notes. This is mainly so because, unlike in other countries (e.g. the United Kingdom), the cashiers of commercial banks, etc. have no instruction to sort the bank notes which are to be returned to the Netherlands Bank, by degree of soiling.

The derivation of formula (3) assumes that all bank notes which have been returned but not rejected are in circulation again in the next period. Thus it ignores the fact that approved bank notes remain at least 2 weeks in storage at the Bank before they are brought into circulation again. This simplification is, however, not essential and quantitatively unimportant.

## 3. Soiling Time and Life of Bank Notes in Seven Circulation Trials

### 3.1. *Circulation Trials*

This paper is based on seven circulation trials. Each trial consists of between one and six series of 100 000 bank notes, which have been brought into circulation simultaneously. The main characteristics of the seven trials are listed in Table 2.

We consider two circulation trials for the Fl. 25 denomination, four for the Fl. 100 denomination and one for the Fl. 1000 denomination. The Fl. 25 bank notes were brought into

TABLE 2
*Characteristics of the circulation trials*

| Denomination | Code | Number of bank notes brought into circulation | First day of issue | Number of weeks involved | Reject percentage |
|---|---|---|---|---|---|
| Fl. 25 | GA | 500 000 | April 5th, 1976 | 195 | 92 |
| Fl. 25 | GB | 600 000 | November 11th, 1976 | 164 | 87 |
| Fl. 100 | MA | 595 000 | July 21st, 1975 | 230 | 78 |
| Fl. 100 | MB* | 575 000 | July 21st, 1975 | 230 | 74 |
| Fl. 100 | MC | 600 000 | November 11th, 1976 | 170 | 49 |
| Fl. 100 | MD | 400 000 | July 26th, 1977 | 135 | 51 |
| Fl. 1000 | RA | 100 000 | January 2nd, 1978 | 112 | 23 |

* Paper without flax.

circulation in April and November 1976; at the time of our computations nearly all had been rejected and withdrawn. The first two trials with Fl. 100 notes started in July 1975, and were used by Koeze (1979) to study the difference in durability between paper with flax (code MA) and paper without flax (code MB). The other two trials with Fl. 100 notes are more recent, and relate to the (at that time) customary paper with flax. The Fl. 1000 series was brought into circulation in January 1978; only 23 per cent have been rejected so far. Unfortunately earlier circulation trials with Fl. 1000 notes cannot be used, because the returned Fl. 1000 notes of older series were kept in storage and were not reissued again for some time.

### 3.2. *Average Soiling Time and Life*

The method of Section 2 has been applied to weekly data for the circulation trials mentioned above. Fig. 1 shows both the estimated probabilities of remaining unsoiled and the probabilities of survival. It is obvious that the curve of remaining unsoiled always lies below the survival curve as the difference relates to soiled notes which have not yet been returned to the Bank, nor subsequently withdrawn from circulation. Thus, of the code GA trial with Fl. 25 notes 7·9 per cent have not yet been withdrawn, while, according to our model, only 3·8 per cent of these notes are still unsoiled.

Since not all the notes in the trials have been returned (i.e. $T^* < T'$), we have, in order to estimate the average soiling time and life, extrapolated the missing data on the assumption of a constant hazard, set equal to the average in the sample period. The resulting average soiling time and life time are listed in Table 3. The table gives these results both for the case where, after 6 years, all notes have been withdrawn, and for the case where the notes have a maximum life of 15 years.

TABLE 3

*Estimated average soiling time and life time (in weeks)*

| | | *Complete withdrawal after 6 years* | | *Complete withdrawal after 15 years* | |
|---|---|---|---|---|---|
| *Denomination* | *Code* | *Soiling time* | *Life time* | *Soiling time* | *Life time* |
| Fl. 25 | GA | 49 | 76 | 49 | 77 |
| Fl. 25 | GB | 63 | 90 | 63 | 91 |
| Fl. 100 | MA | 101 | 140 | 106 | 158 |
| Fl. 100 | MB* | 128 | 165 | 134 | 191 |
| Fl. 100 | MC | 157 | 188 | 187 | 249 |
| Fl. 100 | MD | 130 | 162 | 144 | 197 |
| Fl. 1000 | RA | 203 | 225 | 303 | 366 |

* Paper without flax.

The table shows that large denominations have a longer life than small ones. This is a well-known characteristic of bank notes: notes with a low nominal value circulate faster, and are treated with less care, than notes with a high value, with the result that they are soiled much sooner. For Fl. 100 and Fl. 1000 bank notes, which have a fairly long life, it matters greatly whether we assume complete withdrawal after 6 years or after 15 years. Fl. 25 notes, however, have nearly all been rejected after 6 years. It must be noted here that especially the estimated soiling time and life of Fl. 1000 notes are subject to a wide error margin because only 23 per cent of these notes in the trial have so far been rejected.

Table 3 also shows considerable differences between the outcomes of the respective trials for the same denomination. For instance, the control trial (code MA) of Koeze's experiment

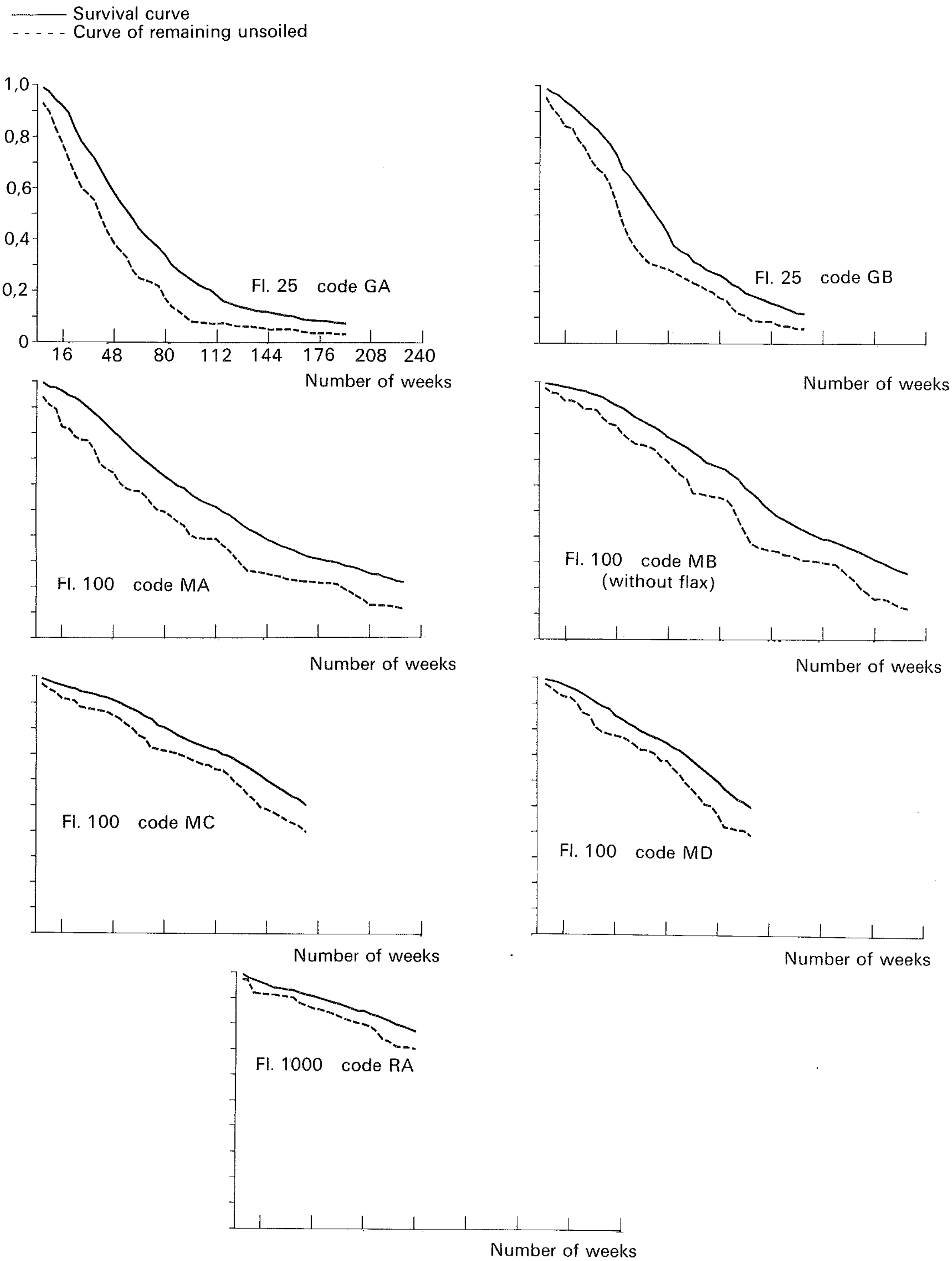


FIG. 1. Estimated probabilities of remaining unsoiled and of survival in the circulation trials. The scale of the axes is the same for all seven graphs.

measuring the influence of a different quality of bank note paper turns out to yield a much shorter soiling time and life than the trial with paper without flax (code MB). As a consequence Koeze prefers paper without flax to paper with flax. The two other Fl. 100 trials (with flax), which were brought into circulation later, however, yield either similar or longer soiling times and life times as Koeze's notes without flax. We will discuss the significance of these differences further on in this paper.

## 4. The Withdrawal Process

Although bank notes are sorted mechanically, the process of withdrawal is subject to fluctuations, i.e. the soiling criterion on which rejection is based, is not always the same, owing to the fact that, among other things:

(a) several machines are used;

(b) the optical instruments, through which the reflected light intensity of bank notes is measured, become dirty and have to be cleaned;

(c) sometimes the machine is readjusted with the result that the soiling criterion is changed.

As a result, the outcomes of the circulation trials do not follow a probability distribution with the statistical accuracy to be expected of such large samples. In other words, the results cannot be considered to be drawings from the same distribution. Consequently, we encounter the following problem in applying the method of Section 2. If the number of returned and rejected bank notes consists of only a very small part of the sample population, as is the case in the weekly figures, formula (3) may be approximated by

$$p_t \approx (N_{t-1}/N)(v_t/o_t - v_{t-1}/o_{t-1}).$$

This formula shows that the rejection fraction $v/o$ must be larger in period $t$ than in period $t-1$. Theoretically, this is of course always the case as bank notes of the same generation become relatively more and more soiled when circulating. In practice, however, this rule does not always hold true in our trials, so that the $p$'s sometimes assume negative values. These incongruities indicate the necessity of graduating or smoothing our trial data using appropriate parametric distribution functions.

### 4.1. *Graduating the Data*

We assume that the process of soiling follows a probability distribution with density function $f(t, \alpha)$, so that $p_t = f(t, \alpha)$ where $\alpha$ (possibly a vector) is to be estimated. In formula (3) $p_t$ is replaced by $f(t, \alpha)$ and $v_t$ by $v_t^e = v_t + u_t$ where $u_t$ is a random component of variation in $v_t$. Thus (3) may be re-written as

$$v_t^e = \frac{o_t}{N_{t-1}} \left\{ f(t, \alpha)\, N + v_{t-1}^e \left( \frac{N_{t-2}}{o_{t-1}} - 1 \right) \right\}. \tag{4}$$

Apart from the fluctuations inherent in the rejection process the random component $u_t$ can also describe other non-recurrent outliers, for example when for some reason a large parcel of soiled bank notes is returned to the Bank. The parameter(s) $\alpha$ of the distribution function can be estimated by minimizing the sum of squares of these components, $\Sigma_{t=1}^{T^*} u_t^2$, with respect to $\alpha$ where $T^*$—as before—is the last week of observation in the trial. This introduction of $u_t$ also offers a criterion for selection of the density function $f(t, \alpha)$ best suited to the soiling process, i.e. that distribution will be chosen of which the standard deviation of the residuals is the smallest.

### 4.2. *Soiling Time and Life*

Once the most suitable distribution has been found for the soiling process and its parameters have been estimated the average soiling time ($l$) is computed as the expectation of this distribution [see (2), $T \to \infty$]. Similarly, the average life is calculated by (1) ($T' \to \infty$), where $p_t^+ = v_t^e/N$ so that, according to (4)

$$l^+ = \sum_{t=1}^{\infty} \frac{o_t}{N_{t-1}} (t-1) f(t, \alpha) + \sum_{t=1}^{\infty} \frac{o_t}{N_{t-1}} \left( \frac{N_{t-2}}{o_{t-1}} - 1 \right) (t-1)\, p_{t-1}^+ \tag{5}$$

with $o_t/N_{t-1}$ the *return frequency* in period $t$. If we assume this frequency to be constant and equal to $R$, it follows from (5) that

$$l^+ = l + (1-R)/R. \tag{6}$$

This result shows that when the return frequency is constant, the difference between the life and the soiling time depends solely upon the return frequency. This difference, which can be regarded as the "extra" life of bank notes as they do not return to the Bank immediately after soiling, decreases when the return frequency increases. Formula (6) provides us with a simple rule of thumb to determine this extra life.

### 4.3. *Standard Deviation of the Parameters of the Soiling Process*

On the assumption that the random components ($u_t$) are independently normally distributed with a constant variance, we can determine confidence intervals for $\alpha$ and test for the equality of values of $\alpha$ in different circulation trials using the log likelihood

$$-T^* \ln \{\Sigma\, u_t^2\}$$

which is a function of $\alpha$.

We note, however, that our assumption of normality is not entirely justified. At the end of the circulation trial far fewer bank notes are returned than at the beginning or in the middle of that trial. The number of rejects (in absolute terms) is also much smaller at that time than earlier on in the trial, so that, on that account, the $u$'s decrease in size. The variance of $u$ may therefore not be constant. Furthermore, it appeared that the $u$'s are sometimes positively correlated, as the fluctuations in the withdrawal process tend to be biased in the same direction during a certain period. We can reckon with this autocorrelation in the likelihood-ratio test (and in the estimation of the distribution parameters), but this has been omitted so far. As a consequence the accuracy of the parameter estimates is overestimated.

## 5. Estimation Results

### 5.1. *Distributions for the Soiling Process*

In principle all distributions which are defined on the interval $0 \leqslant t < \infty$ qualify for describing the soiling process. In this section we present the results of three well-known distributions, the exponential, the Weibull and the gamma distribution. In the case of the *exponential* distribution with parameter $\lambda$, the probability that a bank note becomes soiled in the next period is the same for each period, i.e. it is independent of the number of periods that the note is in circulation. In this case the soiling process is a Poisson process: the number of bank notes that becomes soiled is Poisson distributed. The *Weibull* distribution with parameters $\lambda$ and $\gamma$ applies to technical deterioration processes, where the life of each individual part depends on the weakest link (see, for example, Bury, 1975). As such the distribution may appropriately describe the soiling process of bank notes. The two-parameter *gamma* distribution is the sum of a number ($\gamma$) of identical exponential distributions with parameter $\lambda$, and is often used in the description of processes where deterioration is the result of a number of separate events which each have an equal chance of occurring at any given moment. Obviously one would also think of this statistical model as an adequate description of the soiling of bank notes: notes become soiled after they have changed hands and/or have been folded a number of times. However, this plausible description turns out to be contradicted somewhat by the outcome: in most circulation trials the value of $\gamma$ turns out to lie around one.

Apart from these three distributions, we also tried the log-normal distribution which, gave less satisfactory results (e.g. higher standard deviations). For the code GA circulation trial with Fl. 25 bank notes, we experimented further with a combination of two exponential distributions (Schuhl distribution, see Johnson and Kotz, 1970, p. 224) and with other combinations. In this way the existence of (closed) subcircuits was studied, in which bank notes become soiled according to their specific distribution. The proportionality parameter, i.e. the parameter that indicates which part of the sample becomes soiled according to one distribution and which part according to the other, was estimated simultaneously with the parameters of the distributions. For the Schuhl distribution, with three parameters to be estimated, this came out at about 0·5, while the parameters of the two exponential distributions hardly differed from one another. As these outcomes do not, in any way, point in the direction of such subcircuits this experiment was not continued.

### 5.2. *Parameter Estimates and the "Best" Distributions*

In order to estimate the parameters of the distributions the sums of squares of the disturbances is minimized numerically by a quasi-Newton algorithm (ZXMIN from IMSL) and in the neighbourhood of the minimum, by the direct search method of Hook and Jeeves (see, for example, Kowalik and Osborne, 1968). The resulting standard deviations (i.e. the criterion values) for the three distributions are given in Table 4.

TABLE 4

*Standard deviation of the residuals (in numbers of bank notes per week)*

| *Denomination* | *Code* | *Exponential distribution* | *Weibull distribution* | *Gamma distribution* |
|---|---|---|---|---|
| Fl. 25 | GA | 483 | 377° | 380 |
| Fl. 25 | GB | 1,021 | 826 | 815° |
| Fl. 100 | MA | 374 | 364 | 316° |
| Fl. 100 | MB* | 803 | 492° | 503 |
| Fl. 100 | MC | 583 | 475° | 499 |
| Fl. 100 | MD | 751 | 692° | 701 |
| Fl. 1000 | RA | 122 | 115 | 115° |

* Paper without flax.

Explanation: ° indicates the lowest standard deviation.

From Table 4 we may infer a slight preference for the Weibull distribution over the gamma distribution. The exponential distribution yields the highest criterion value throughout. This is evident, as both the Weibull and the gamma distribution are generalizations of the exponential distribution.

Table 4 shows remarkably large differences in criterion value among the trials, even given the differences in sample size. This means that each circulation trial has specific characteristics (manner of issue, date and place of issue, and paper quality), which thwart the comparability of the trials to some extent.

The parameter estimates of the three distribution functions of the soiling process are shown in Table 5. It appears that the value of $\gamma$, determining the shape of both the Weibull and the gamma distribution, exceeds unity in five out of the seven trials investigated.

Fig. 2 shows the probabilities of remaining unsoiled and the probabilities of survival which ensue from the best fitting graduating curve for the seven trials. We may regard these curves as smoothed versions of the directly estimated probabilities of Fig. 1.

TABLE 5
*Parameter estimates of the soiling distributions*

| Denomination | Code | Exponential distribution $\lambda$ | Weibull distribution $\lambda$ | Weibull distribution $\gamma$ | Gamma distribution $\lambda$ | Gamma distribution $\gamma$ |
|---|---|---|---|---|---|---|
| Fl. 25 | GA | 0·0206<br>(0·0003) | 0·0116°<br>(0·0006) | 1·15°<br>(0·01) | 0·0272<br>(0·0008) | 1·27<br>(0·03) |
| Fl. 25 | GB | 0·0149<br>(0·0004) | 0·0053<br>(0·0005) | 1·25<br>(0·04) | 0·0230°<br>(0·0010) | 1·45°<br>(0·05) |
| Fl. 100 | MA | 0·0093<br>(0·0001) | 0·0119<br>(0·0009) | 0·94<br>(0·01) | 0·0082°<br>(0·0003) | 0·91°<br>(0·03) |
| Fl. 100 | MB (without flax) | 0·0059<br>(0·0002) | 0·0005°<br>(0·0001) | 1·53°<br>(0·03) | 0·0146<br>(0·0007) | 2·02<br>(0·08) |
| Fl. 100 | MC | 0·0045<br>(0·0001) | 0·0009°<br>(0·0002) | 1·34°<br>(0·04) | 0·0077<br>(0·0005) | 1·47<br>(0·07) |
| Fl. 100 | MD | 0·0063<br>(0·0003) | 0·0016°<br>(0·0005) | 1·30°<br>(0·07) | 0·0100<br>(0·0010) | 1·40<br>(0·10) |
| Fl. 1000 | RA | 0·0032<br>(0·0002) | 0·0057<br>(0·0008) | 0·86<br>(0·03) | 0·0021°<br>(0·0003) | 0·84°<br>(0·04) |

Explanation: standard deviations are given in parentheses; ° indicates the distribution with the best fit.

### 5.3. *Soiling Time and Life according to the Distributions*

The average soiling time calculated with the parameter estimates of Table 5 is given in Table 6. Contrary to the direct estimates in Section 3, no explicit assumptions are necessary about the tail of the distributions. The table also gives the average life, which has been computed by means of (6) on the assumption of a constant return frequency. For this we use

TABLE 6
*Average soiling time* ($l$) *and life* ($l^+$) (*in weeks*)

| Denomination | Code | Average return frequency ($R$) | | Exponential distribution | Weibull distribution | Gamma distribution |
|---|---|---|---|---|---|---|
| Fl. 25 | GA | 0·031 | $l$<br>$l^+$ | 49<br>80 | 46°<br>77° | 47<br>78 |
| Fl. 25 | GB | 0·038 | $l$<br>$l^+$ | 67<br>92 | 61<br>86 | 63°<br>88° |
| Fl. 100 | MA | 0·024 | $l$<br>$l^+$ | 108<br>149 | 111<br>152 | 111°<br>152° |
| Fl. 100 | MB (without flax) | 0·027 | $l$<br>$l^+$ | 168<br>204 | 133°<br>169° | 138<br>175 |
| Fl. 100 | MC | 0·035 | $l$<br>$l^+$ | 222<br>250 | 179°<br>207° | 190<br>218 |
| Fl. 100 | MD | 0·039 | $l$<br>$l^+$ | 158<br>183 | 132°<br>157° | 140<br>165 |
| Fl. 1000 | RA | 0·038 | $l$<br>$l^+$ | 317<br>342 | 446<br>471 | 398°<br>423° |

Explanation: ° indicates the distribution with the best fit.

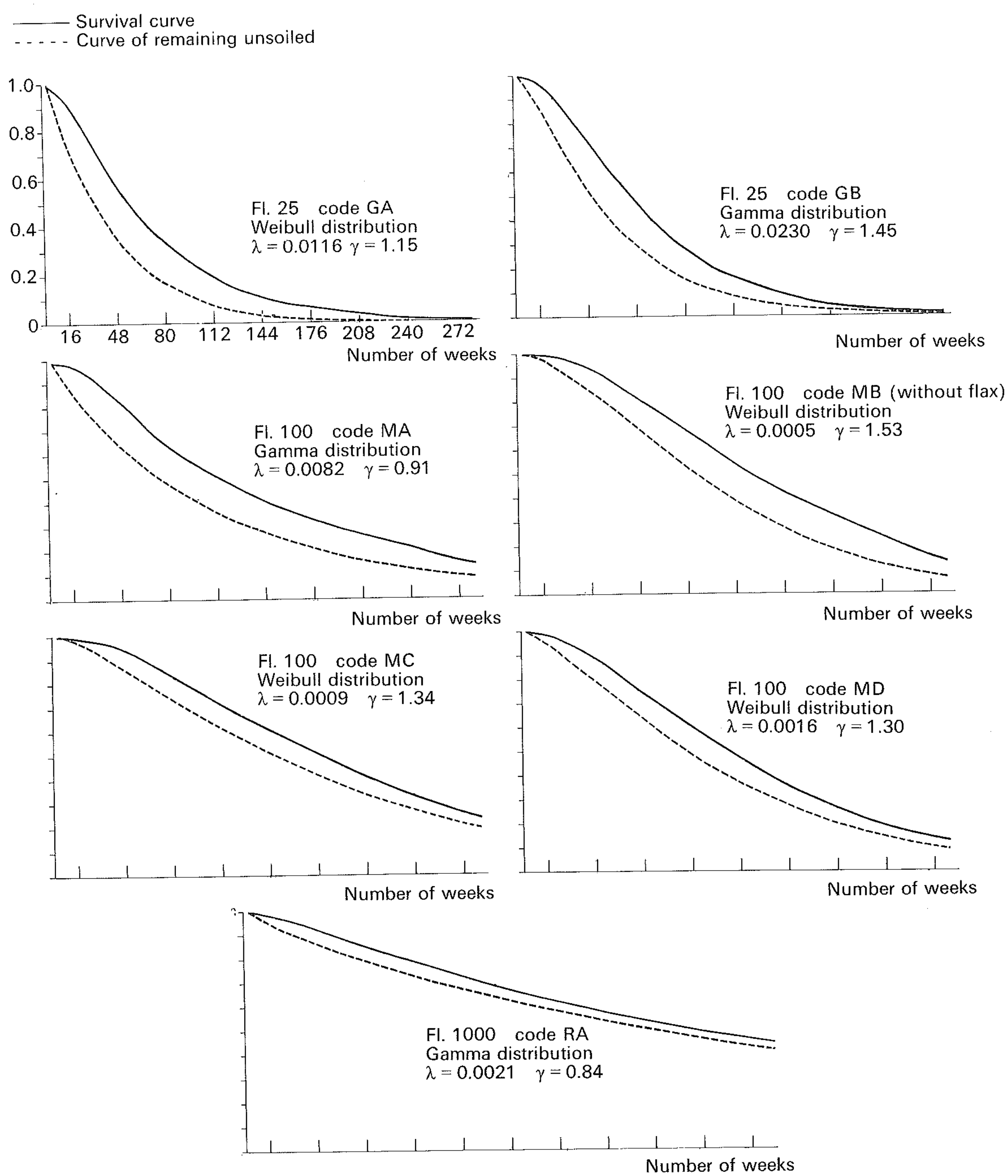


FIG. 2. Probabilities of remaining unsoiled and probabilities of survival according to the best fitting graduating curves. The scale of the axes is the same for all seven graphs.

the average frequency in the sample period. We note that the return frequency in most trials is not constant, but decreases somewhat. An explanation for this phenomenon could be that in the course of time an increasing number of bank notes land in circuits, from where they seldom or possibly never return to the Bank. It is remarkable, however, that the decline in the return frequency of Fl. 1000 notes does not exceed that of Fl. 100 and Fl. 25 notes, so that "underground" circuits cannot form an explanation. In calculating the average time between soiling and withdrawal, it would be possible to make allowance for a decreasing return frequency in (5), but so far this has been omitted.

We can see from Table 6 that the measured average soiling time (and thus the life) of the Fl. 25 notes hardly depends on the choice of the distribution function, nor do the results deviate much from those found through direct measuring and extrapolation (see Table 3). However, considerable differences occur between the results of the two trials. This phenomenon is even more apparent in the trials with Fl. 100 notes. Koeze's control trial with paper containing flax (code MA) yields an average soiling time of slightly over 100 weeks, whereas the MC coded trial comes out at an average soiling time of nearly 200 weeks. Moreover, for the Fl. 100 notes, and also for the Fl. 1000 notes, the outcomes depend much on the distribution function used. In this respect we note that for the Fl. 100 notes, with the exception of the MA coded trial, a much longer soiling time is estimated with the exponential distribution than with the Weibull and gamma distributions. The opposite is true for the trial with the Fl. 1000 notes, where the considerable differences in outcomes may be partly due to the fact that the trial has only been going on for a (relatively) short period.

Nevertheless, Table 5 shows that the standard deviations of the coefficients are fairly small for the Fl. 1000 notes trial. It is possible that they are favourably influenced by the fact that Fl. 1000 notes are not sorted every week, so that we had to insert fictional observations (1 bank note received, 0 rejected).

It is remarkable that there are no systematic differences between the various denominations with regard to the average return frequency.

### 5.4. *Test of Equality of the Parameters*

In Table 7 the hypothesis of equal parameters of the (same) distribution functions is tested for pairs of trials of the same denominations. The null-hypothesis assumes that the process of soiling is independent of the date on which the notes are brought into circulation. In the case of the exponential distribution rejection of this hypothesis implies that the differences in soiling time are significant.

The table shows that in all cases but one the null-hypothesis is clearly rejected. The exception relates to the parameter of the exponential distribution for the MB coded series without flax and the MD coded series with flax. It must be noted that the parameters of the

TABLE 7
*Likelihood-ratio test of the hypothesis of equal parameters, in pairs of trials*

| *Denomination* | *Trial 1 code* | *v.* | *Trial 2 code* | *Exponential distribution* | *Weibull distribution* | *Gamma distribution* |
|---|---|---|---|---|---|---|
| Fl. 25 | GA | | GB | 85·1 | 145·2 | 148·6 |
| Fl. 100 | MB* | | MA | 133·4 | 371·0 | 363·6 |
| Fl. 100 | MB* | | MC | 34·4 | 124·3 | 127·4 |
| Fl. 100 | MB* | | MD | $0{\cdot}96^{\mathrm{nr}}$ | 10·7 | 16·4 |

* Paper without flax.

Explanation: The test value is $\chi^2$ distributed with 1 degree of freedom in the case of the exponential distribution and 2 degrees of freedom in the case of the other distributions; nr indicates not rejected.

Weibull and the gamma distributions do differ significantly in these two trials, despite the fact that the two estimated soiling times are virtually the same (see Table 6).

The distribution parameters of the two circulation trials used by Koeze (1979, 1980) differ very significantly from one another (code MB v. MA, second line in Table 7) and confirm Koeze's results that the bank notes without flax become soiled less rapidly than the bank notes with flax of his control trial. On the other hand, the notes without flax become soiled much faster than the MC coded series with flax (third line of Table 7). These marked differences between the various trials are surprising; therefore we think it advisable to base a study concerning bank note paper durability on a number of circulation trials and control trials, not solely on a single pair.

## 6. Conclusions

This paper describes the processes of soiling and withdrawal of bank notes. It determines the best fitting probability distribution for the soiling process and measures the average soiling time and life of bank notes, for a number of trial series.

The Fl. 25 notes in the two trials considered turned out to become soiled after 1 year on average. It takes another 25–30 weeks until the soiled notes are returned to the Bank and withdrawn from circulation. Thus according to these trials the average life of Fl. 25 notes is over $1\frac{1}{2}$ years.

We investigated four different circulation trials of Fl. 100 notes. The average soiling time found here is between 2 and 4 years. The average time betweeen soiling and withdrawal is 25–40 weeks, so that Fl. 100 notes turn out to have an average life of $2\frac{1}{2}$–$4\frac{1}{2}$ years.

The calculations for the Fl. 1000 notes are based on one trial series only. Dependent on the probability distribution of the soiling process, we find an average soiling time of 6–8 years and an average life of $6\frac{1}{2}$–$8\frac{1}{2}$ years. These results are, however, subject to a wide error margin; on the basis of other information not presented in this paper they seem rather on the high side.

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